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Resumo Mathematics is about relationships, repeatability, and nested structures. Regular ordering affects us viscerally because human perception relies upon information reduction through symmetries. Random (disorganized) information becomes too much for us to process, which generates anxiety. Architectural elements are visible shapes, and they need to be combined, compared, counted, grouped, and juxtaposed. This is what our brains do automatically. We subconsciously analyze and process the information ...

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HOW MATHEMATICS CREATES MEANING IN ARCHITECTURE

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Abstract: Mathematics is about relationships, repeatability, and nested structures. Regular ordering affects us viscerally because human perception relies upon information reduction through symmetries. Random (disorganized) information becomes too much for us to process, which generates anxiety. Architectural elements are visible shapes, and they need to be combined, compared, counted, grouped, and juxtaposed. This is what our brains do automatically. We subconsciously analyze and process the information presented in any composition using mathematical relations. We perceive our world by grouping adjoining geometrical elements, via symmetries, into larger wholes. We make our way in the world thanks to a mathematical process for making sense of our environment. Basic symmetries have a profound effect on composition and design. Some elements have the same size and shape (oriented in the same way, or reflected, or rotated), and are aligned horizontally or vertically. Their repetitions are regularly spaced; otherwise there is no symmetry. Scaling symmetry is something entirely distinct, and links components visually when we see magnified or reduced versions of the same thing. This self-similarity at different magnifications is a basic feature of a 'fractal'. Scaling symmetry is a dominant feature in traditional and vernacular architectures, and is one reason those quite different form languages have meaning for us. Mathematics also relates components of a whole via their relative number and size. The universal distribution law says: "In a complex system, there are few large objects, more intermediate-size objects, and many smaller objects, roughly in an inverse-power relationship". The number of elements of different sizes we perceive at the same time should be inversely proportional to their size. These requirements influence architectural composition to have an "ordered" appearance that echoes traditional and vernacular styles.

Keywords: Architecture; Mathematics; Meaning; Order; Symmetries.

1. Introduction

Mathematics has historically been tied to architecture as one of its most important tools. In many eras that produced great architecture, an architect combined the professions of architect/mathematician. Architects and structural engineers need mathematical knowledge to make a building stand up. Another side of mathematics relates aesthetic expression to overall form and tectonics. For example, proportional ratios were used to determine relative dimensions of architectural components. Yet in contrast to the down-to-earth applications of mathematics to the engineering of buildings, the aesthetic part is full of mystery and romanticism — unlike what mathematics is actually about.

These questions surpass architecture *per se* and immediately broaden to mathematical definitions of beauty. Can we define formulas for beauty? It is very difficult to do so, although this has not stopped many authors from trying. The results are mixed: at best confusing and of doubtful practical value. Yet there are positive examples, and how do we explain those? What appears to be the case is that the neurological intuition of the designer underlies whatever mathematical method is being used. One case is the much-discussed "Golden Ratio" or "Golden Mean", which I'm sorry to say does not endow any magical or special meaning to rectangles (Salingaros, 2018).

The key to understanding the role of mathematics in design is to discover how mathematical patterns in nature became embedded into our neurological systems. "Beauty" then represents those hard-wired patterns we respond to for reasons that guaranteed survival during our evolution. By investigating those original natural patterns, we come up with a set of basic mathematical tools (Salingaros, 2010; 2019). Notions of entropy, fractals, information compression, simple transformations in a plane, and vector spaces arise as the most relevant tools for assessing architectural compositions.

Intuitive beauty summarizes our evolved computational algorithms for survival in an informational environment. Beauty attracts us because we subconsciously interpret it as nourishing; it's the special type of mathematically-ordered complexity that heals us. "Alien" shapes in our immediate surroundings are the opposite of beautiful. Because they do not remind us of natural shapes that our evolution has programmed us to interpret, they disturb us. Alarm and the "fight or flight" response take over our body until we either have enough information to judge that some object is harmless, or decide to flee.

There is more to beauty than utilitarianism: complex recurring patterns are found in inanimate physical structures in the universe; hence some of our key notions of beauty originate with the structure of matter itself (Alexander, 2001). This is physics, not biology. It cannot possibly have anything to do with evolutionary adaptation, because it goes far deeper and was defined before life

evolved. A geometrical necessity for structural coherence is built into our body, and co-exists with separate aesthetic instincts arising from what is biologically "useful". A visceral kind of beauty is independent of anybody's opinion.

2. The Golden Mean and the Fibonacci sequence.

What is called the Golden Mean (or Golden Ratio) is an irrational number approximately equal to 1.618 and usually denoted ϕ (the Greek letter Phi). This number arises as the solution to the problem of subdividing a rectangle into a square x^2 and a remaining, smaller rectangle that is similar to (i.e. has the same aspect ratio as) the original large rectangle (Figure 1).

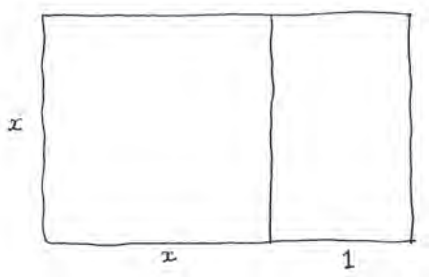

Figure 1. *A Golden Mean rectangle in which the piece "left over" after defining a square has the same aspect ratio as the original rectangle.*

The geometrical problem is described by the relation $(x + 1)/x = x$, leading to the equation:

$$x^2 = x + 1 \qquad (1)$$

Equation (1) has the positive exact solution $x = \phi = (1 + \sqrt{5})/2$, which is the Golden Mean. The Golden Mean is linked to the Fibonacci sequence. Consider the sequence of positive integers $\{ a_n \}$ defined by the recursion relation $a_{n+2} = a_{n+1} + a_n$, with $a_1 = a_2 = 1$, giving:

$$\{ 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ... \} \qquad (2)$$

Starting from 1 and 1 generates the entire sequence, and is the simplest way to describe growth by adding two numbers to obtain the next one.

One can obtain rational approximations of the Golden Mean ϕ with ever increasing accuracy from the ratios of consecutive numbers in the Fibonacci sequence Equation (2). This result is due to the great astronomer Johannes Kepler (1619). Listing some of these ratios gives the following approximations:

$$\{ \ldots 8/5 = 1.60, 13/8 = 1.625, 21/13 \approx 1.615 \ldots \}$$

Therefore, the ratio of successive terms in the Fibonacci sequence Equation (2) tends to the Golden Mean ϕ in the limit:

$$a_{n+1} / a_n \rightarrow \phi = 1.618...$$

3. Hierarchical subdivisions and scaling.

Any serious theory of architectural design ought to describe hierarchical subdivisions, scaling, and the relationship among distinct scales. I propose that a building should have well-defined subdivisions at dimensions that correspond to powers of e , equal to 2.718… (Salingaros, 1998; 2010). The design method uses the largest dimension L of the structure, making sure that substructures exist very roughly at L/e , L/e^2 , L/e^3 , etc. all the way down to the size of the granulation in the materials themselves (see Figure 2, below). The dimension (size) of components at each of these levels of scale is approximate; what is crucial is that no level of the hierarchy should be missing.

Design is linked mathematically with natural growth through hierarchical subdivisions at distinct scales, which is found in a majority of natural structures. There is furthermore a regular geometrical relation among different scales of substructure, and in many cases, the scales are related by a single scaling factor. This theory is based on systematic observations and measurements by Christopher Alexander (2001), who found that scaling factors of around 3 (with an extended range roughly between 2 and 5) tend both to predominate in nature, and to be preferred by human observers.

The scales in a natural hierarchy are skewed towards the smallest sizes. Growth begins at the infinitesimal scale and develops through an ordered hierarchy up to the largest size. The spacing of different scales is therefore not uniform. There are proportionately more small levels of scale than large scales (Alexander, 2001; Salingaros, 1998; 2010), something that is not obvious from discussions about the size of the scales themselves. I will explain this later.

If one wants to recast this scaling theory as a sequence of integer factors so as to compare it to the Fibonacci sequence Equation (2), then successive powers of e can be rounded out to:

$$\{ 1, 3, 7, 20, 55, 148, 403, 1097, ... \} \quad (3)$$

This sequence makes more accurate an old prescription sometimes used in traditional design: "*subdivide everything by three*" (Figure 2). Of course, all that a designer needs is to repeatedly divide by $e \approx 2.718$, and most pocket calculators have e built in. The numbers in the sequence (3) have no intrinsic importance: they simply approximate an exponential sequence of scales by integers for the purpose of comparing with Equation (4), below.

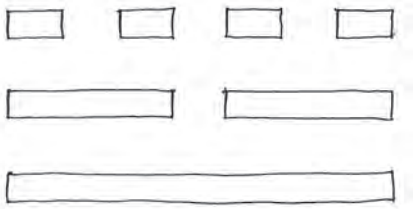

Figure 2. *Components of a complex structure have, for example, 4 pieces of size 1, 2 pieces of size 3, and 1 piece of size 7.*

To clarify the point about scales being distributed more towards the smaller end of the spectrum, let's generate a set of measures as a guide for some design project, beginning from the smallest perceivable detail at, say, 0.5 mm. Multiplying repeatedly by the scaling factor e gives the following example sizes, where the numbers are rounded off:

{ 0.5 mm, 1 mm, 4 mm, 10 mm, 3 cm, 7 cm, 20 cm, 55 cm, 1.5 m, 4 m, 11 m }

In actual design, the brief and human dimensions fix the larger scales, then the smaller scales are computed from those: here we worked in the opposite direction — from small to large — in order to illustrate the theory. These measurements may be useful or not depending upon whether the larger sizes correspond closely to what a particular design requires. Note that in a structure of 11 m size, there are eight scales smaller than, and only three scales larger than 1 m. The smaller scales are much "tighter". This is a key to understanding the enormous discrepancy between traditional and modernist design (Alexander, 2001; Salingaros, 1998; 2010).

The sequence of integer approximations to the powers of e in Equation (3) compares very roughly to alternate terms of the Fibonacci sequence Equation (2). That means that the even terms of the Fibonacci sequence Equation (2) could, if desired, be used for the theory of design based upon a scaling hierarchy:

$$\{ 1, 3, 8, 21, 55, 144, 377, 987, ... \} \quad (4)$$

The numbers in Equation (4) can be generated as a sequence { b_n } with recurrence relation $b_{n+2} = 3b_{n+1} - b_n$, $b_1 = 1$, $b_2 = 3$. In the limit, ratios of consecutive terms of Equation (4) tend to a number $\psi = 2.618\ldots$ (the Greek letter Psi), which is the positive solution of the equation $x^2 = 3x - 1$. This number ψ is related to the Golden Mean as $\psi = \phi^2 = \phi + 1 \approx 2.618$, and notice that ψ is within 4% of the value of e . All of this discussion attempts to make more useful Alexander's original findings of scaling by a factor anywhere from 2 to 5 (Alexander, 2001).

Design can thus be guided by knowing a sequence of sizes that should be defined very approximately by the tectonics of the structure itself. Where structural members don't provide a required scale, the architect creates ornament. This is a key point. To the best of my knowledge, architects have never consciously implemented this tool (other than sometimes applying the "rule of 3"), yet we universally find built examples with such subdivisions. What probably happened is that builders throughout history simply created subdivisions that "felt right" because those mimicked natural forms.

The nested rectangles shown in Figure 3 generate a hierarchy of scales — not ratios of sides — that can then be used to approximately regulate a structure's subdivisions. A second point is to recognize fractal scaling, where similar components (four "golden" rectangles in this simplified case, but in practice any shape at all) repeat at different magnifications. Scaling similarity is the main characteristic of all fractals, and can be found in many of the world's most beloved historical buildings (Alexander, 2001; Salingaros, 1998; 2010).

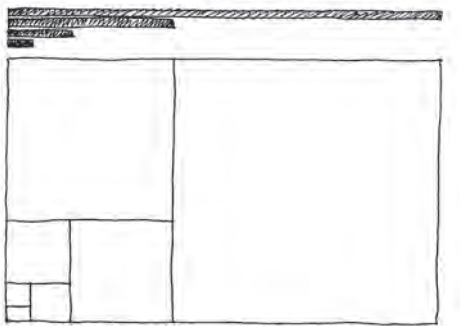

Figure 3. *One could equally well use nested Golden Mean rectangles to generate the alternate Fibonacci numbers, which then define the sizes in a natural scaling hierarchy.*

The relative lengths at the top of Figure 3 correspond to the numbers in the sequence Equation (4). Why do we want to use only every other term of the Fibonacci sequence? The reason is that we wish to measure and compare the size of design components using a *scaling ratio*, and not compare sides of a rectangle that define an *aspect ratio*.

To summarize: the Golden Mean ϕ is useful in human creations in the same way it is found to occur in nature, where it related to the hierarchical scaling that is a consequence of organic growth. A natural hierarchy of scales can be generated by either an exponential or a Fibonacci sequence, which provide a checklist of component sizes.

The design method just described assumes that built structures will be approximate, and thus allows for a wide tolerance and considerable deviations from the numbers given above. Thus, in real-world design, one creates an approximate hierarchy of different scales in trying to mimic natural growth as best as possible: not through precision, but through hierarchical complexity. The simplistic application of Golden Mean rectangles can lead to minimalist rectangles, which represent the opposite effect to what I am presenting here.

4. Universal (fractal) distribution of sizes

Mathematics relates components of a whole via their relative number and size. The universal distribution law (Salingaros and West, 1999) says: "In a complex system, there are few large objects, more intermediate-size objects, and many smaller objects, roughly in an inverse-power relationship". This means that the number of elements of different sizes we perceive at the same time should be inversely proportional to their size (see Figure 2, above). More-refined versions of this law follow a scaling index that corresponds to the fractal dimension, and is not simply equal to –1. The components of a fractal are all self-similar through scaling (an additional relationship), which gives it its geometrical coherence.

Let p_i be the number of design elements of a certain size x_i . Then, the number of elements of each size is inversely proportional to their size, where the constant C is fixed by the largest size, and the power m corresponds to the fractal dimension.

$$p_i = C/(x_i)^m \qquad (5)$$

In a frequency distribution, sizes x_i are measured as lengths, whereas multiplicities p_i are integers. Mathematical fractals are generated as an infinite series of scaled-down copies of a single element, and illustrate ideal cases of this law. For the Sierpinski gasket, where the area of a triangle gets progressively subdivided into smaller and smaller triangles, the scaling factor is 2 and the scaling index equals the fractal dimension $m = D = \ln 3/\ln 2 \approx 1.58$. For the von Koch snowflake, where the sides of a triangle get progressively subdivided into smaller and smaller triangles, the scaling factor is 3 and $m = D = \ln 4/\ln 3 \approx 1.26$ (Salingaros and West, 1999). The easiest and most practical choice in architectural applications is $m = 1$.

The inverse-power law Equation (5) is derived from proportionally distributing entropy among all the available scales in a complex structure (Salingaros and West, 1999). It is also related to the allometric growth law satisfied by many natural and especially biological systems.

A fundamental mathematical requirement governs subdivisions of a design or structure, and a scaling law checks those different scales. In addition, we have derived a numerical distribution of objects or components on each scale. This measure applies to fractals. The universal distribution is independent of simple geometric shapes and leads to the coherent structures found in the plant world, where nothing is truly straight. Artificial complex systems also evolve toward such a distribution as they acquire "emergent properties". Examples include electrical power grids, ecosystems, Internet links, and the structure of languages (Zipf's Law).

The universal (fractal) distribution lies at the basis of human perception. Details on the smaller scale establish the *meaning* of what we see. Contours, sharp details, and edge features (high spatial frequency) are more important than larger shapes (low spatial frequency) for interpreting visual information of a complex scene. Clinical *fMRI* studies of brain responses reveal that "representations of scene content are also more strongly conveyed by high than low spatial frequencies" (Berman *et al.*, 2017). This is true even though the global forms are processed first.

Coincidentally, linking high spatial frequencies to an image's meaning explains why a line drawing can capture the character and expression of a person in a portrait. The opposite — minimalist design — eliminates all but the largest shapes, which is the cognitive equivalent to blurring an image (loss of all higher spatial frequencies).

There is a very different concept of addition as spatial grouping to generate a larger element that consists of repeated units (spatial periodicity). Copies of the same element along an axis generate a sequence with repeating elements; but without intermediate grouping it becomes difficult to relate the whole to its numerous smaller components. Such a structure disturbs our cognitive experience because of the gap in scales. Monotonous repetition is perceived as unnatural, and makes us uncomfortable because it breaks the fractal distribution law (Salingaros, 2011).

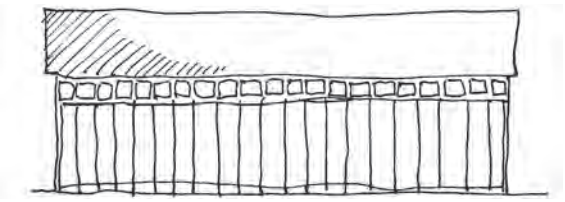

Figure 4. *Monotonous repetition violates the fractal distribution law by eliminating intermediate scales.*

Our neurophysiology interprets spatial data by performing something akin to Fourier or wavelet decomposition. Individual brain cells in the visual cortex perceive different spatial frequencies and orientations, which are then combined (DeValois and DeValois, 1988). There are two lessons here for perception. (*i*) A single spatial frequency depends upon regularity, i.e. translational symmetry. Departures such as irregular spacings are felt as visually dissonant because their encoding requires more neural processing. We instantly notice a spacing that is off, because it creates disjoint regular sequences. (*ii*) Scenes with one predominant spatial frequency are trivial. Since the brain is designed to analyze complex information having the full range of spatial frequencies it evolved to handle in natural environments, we perceive that something is missing.

5. Counting and grouping reduce informational overload

Even the simplest mathematical notions turn out to be very important for how we perceive our environment. In a virtual operation, the observer compares one geometrical component with another to check whether they match or not. Design elements could be counted when they have more or less the same size, shape, and orientation. Redundancy and similarity of shape reduce information overload. The brain then treats them as copies of the *same* element. If, on the other hand, dissimilar elements appear in a composition, they need to be accounted for individually, which takes up information processing in our brains that is needed for other life tasks.

This is not the end of the story, however. The mere presence of several copies of the same element can still lead to information overload, if their positions are unrelated. Symmetries in position reduce this extra information needed to fix the location of elements distributed in space into a more manageable amount. Visual techniques for doing this use multiple symmetries to form each group, and also define a wide border to contain a group (Alexander, 2001).

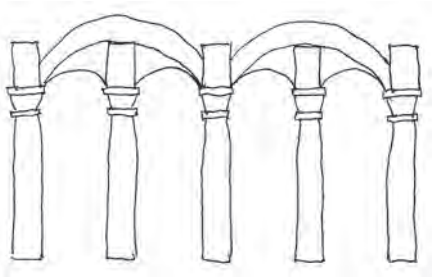

Figure 5. *Arches group three columns into one repeating unit.*

Information coming from numerous unaligned copies of the same repeating element is difficult to grasp, since our sensory system doesn't actually count, but perceives numbers visually as patterns (except for some autistic persons who can instantly count a large number of randomly-distributed objects). Psychologically, this effect is known as the "cognitive limit of 7", which is the maximum number of easily remembered digits, such as a phone number. A large number of elements can be better handled cognitively by grouping them, so that we count the groups instead of the smallest individual elements.

Cognition includes the mechanism of "perceptual invariance", which matches a pattern after displacement, rotation, or scaling of the original. A similarity distance between two visual elements is the number of transformations needed to get from one to the other. The shortest measure equals 1 in the case where an exact copy is displaced by some physical distance. Scaling of concentric figures again equals 1, whereas scaled-up or scaled-down copies at some separation count as 2 (displacement plus the scaling transformation). Our cognitive/physiological system is not as general as the mathematics, however, since it privileges the vertical axis.

For repeating design or structural elements to be aligned, they need to have at least one common dimension, with possible variations. Either the repeating elements are regularly spaced, or they are grouped into a more complex unit, which then repeats regularly. Grouping components into a larger perceptual whole fixes the next scale up in the hierarchy. A tilted copy, however, is not subject to the same simple grouping as are those elements with strict translational symmetry. The increased similarity distance of tilted figures counts as a weaker correlation.

Scaling symmetry is something entirely distinct from the other types of positioning symmetry, and links components visually when we see magnified or reduced versions of the same thing. This is the basic feature of a fractal (think of a cauliflower), which contains a large number of substructures, all of which are self-similar at different magnifications. Scaling symmetry is a dominant feature in traditional and vernacular architectures, and is one reason those quite different form languages appeal to our innate mathematical sense.

6. Reflectional, rotational, and translational symmetries

Architectural elements as visible shapes need to be combined, compared, counted, and juxtaposed. Very generally, combinatorics and relations are derived from *adding*, *aligning*, *counting*, *grouping*, and *repeating* the elements in a composition. These design operations, usually performed subconsciously, trigger our aesthetic response.

A general mathematical-visual mechanism by which we interpret our world is to group adjoining geometrical elements via symmetries into larger wholes. For example: (*i*) two juxtaposed mirror images are joined to make a symmetric whole; (*ii*) aligned repetitions of the same element are joined to make a larger whole having translational symmetry; (*iii*) juxtaposed elements that are related via rotation can be grouped into a larger round whole. Symmetric relations order our environment; they also work at a distance, although their strength decreases. We can combine elementary symmetries: for example, translation with reflection into what is known as a 'glide symmetry'.

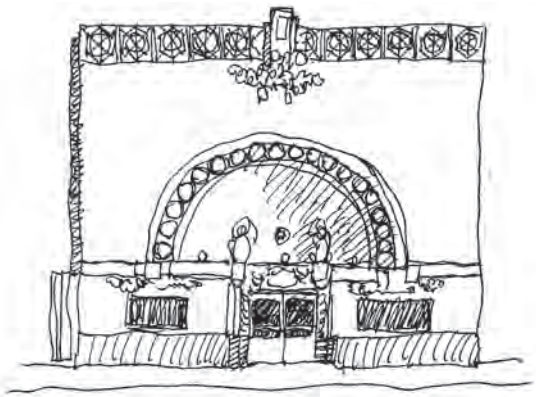

Figure 6. *Rotational symmetry with repeating elements focuses on an entrance.*

To implement translational symmetry in composition, design elements need to have the same size and shape (either oriented in the same way, reflected, or rotated), and be aligned horizontally or vertically. Their repetitions should be regularly spaced: otherwise there is no positioning symmetry. These minimal requirements influence architectural composition to have a certain "ordered" appearance, one that echoes traditional and vernacular styles. Yet this comes from mathematics — supplemented by neuroscience working with physics, which privilege the horizontal and vertical symmetry axes because of gravity — and determines architectural "style".

7. Privileging the vertical axis

Mathematical notions of "beauty" correspond to what is favored by our sensory system. We are constantly processing information in our immediate environment, comparing and looking for groupings, a task that consumes a lot of metabolic energy. We are often overloaded with environmental information, and

rely on built-in algorithms to reduce and organize it. If we cannot instantly classify and categorize forms and shapes surrounding us, we continue to process the information indefinitely, which tires us. This is known as "cognitive fatigue".

Of the two means to handle an overwhelming amount of information by actively shaping our surroundings — (*i*) eliminate it, or (*ii*) organize it — only the second option endows *meaning* to the environment. Unfortunately, dominant architectural culture adopted the first option universally ever since the 1920s. It failed to consider human interface requirements. As a result, minimalist environments have no mathematical meaning, and their creators have had to invent intellectual constructs to take its place. But the human body reacts consistently and perceives such places as meaningless (Mehaffy and Salingaros, 2011; 2013).

Neurophysiology supports this line of reasoning because specific brain cells are designed to recognize shapes and symmetries. Individual neurons respond to specific colors, simple geometric shapes, distinct orientations (angles), and some rather complex shapes essential to our evolutionary survival. Among the latter are "face-recognition" cells, which respond to bilateral symmetry about a vertical axis, and to a generic facial structure of "mouth" with two "eyes" above (Sussman and Hollander, 2015). Our brain is wired to recognize symmetric combinations of simpler elements into more complex wholes.

Adding a strict neurological constraint, our inner-ear mechanism controlling balance prefers a vertical axis. Consequently, diagonals could and do trigger nausea in the observer. This is the reason why, during millennia, symmetry axes never departed from the vertical, and if they did so by accident (such as in the leaning Campanile of the Cathedral of Pisa, Italy), the result became notorious.

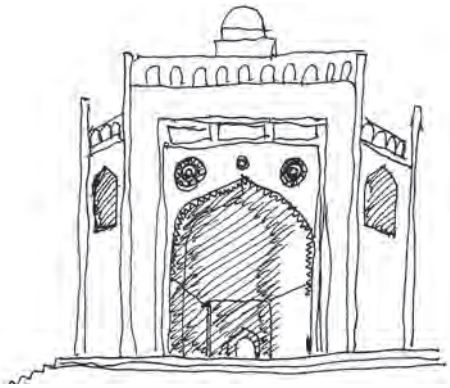

Figure 7. *Bilateral symmetry in a building that is reminiscent of a "face".*

Violating the vertical axis, and neglecting reflectional symmetry about a vertical axis, creates anxiety in the viewer. Building façades that lack such bilat-

eral symmetries either repel us or simply do not register, even as we look directly at them (Sussman and Ward, 2017). Furthermore, if a building's entrance is not marked using our innate preference for a symmetrical, face-like design, it's easy to miss. In general, a composition must employ scaling and bilateral symmetries to focus on the entrance to the building. Deliberately avoiding this cognitive rule compromises so many buildings built since the end of World War II, where the design style conspires to hide the entrance.

8. Vector spaces and closure

Through the process of ordering our environment cognitively, the human brain created Mathematics (Lakoff and Nuñez, 2001). Seeking patterns of coherence and consistency, classification simplifies informational disorder and leads to a logic of classes. It is reassuring and satisfying to know what belongs inside a class — defined according to relational rules — and what remains outside. Experiencing an environment where its elements obey a closure property rather than random elements randomly distributed endows it with meaning. Closure in this sense satisfies the brain's informational need for regularity relations among elements of mathematical classes that we confront in our daily lives.

A Vector Space is a collection of objects that can be added and have scaling (i.e. one can magnify each element proportionally). Notions related to a Vector Space correspond in a deep way with our perceptive apparatus. We like to relate the components of what we see in two ways: (*i*) compare them visually at a distance to determine whether two or more components could be grouped by similarity; and (*ii*) link components by scaling, where we see magnified or reduced versions of the same thing.

Design that incorporates advanced cognitive capabilities is related to a more specialized mathematical framework than a vector space. Since what is presented here is an intuitive investigation, it necessarily mixes mathematical notions with mechanisms of human perception. The treatment of architectural elements relies to a large extent on techniques used in computer graphics. It is useful to identify the visible components of a building or space with vectors in an abstract sense. Two operations, vector addition and multiplication by a scalar, are used here by analogy. Recall the four main axioms for a vector space:

1. The zero (null) vector is in the space.
2. Closure under addition.
3. Closure under reflection.
4. Closure under scaling.

The zero vector must be something that is perceived as nothing at all. Candidates for the architectural "zero vector" are flat white or gray walls, plate glass, curtain walls, or purely reflective surfaces. Perfectly smooth white, transparent, or totally reflective surfaces do not offer the human eye anything to focus on, and therefore the brain interprets that there is nothing there. Already there is a problem of how to handle the possibility of three tectonically distinct types of zero vector: colorless flat, transparent, or reflective. I do not attempt to answer this question here, but note with alarm that those are the preferred architectural surfaces since the 1920s.

A first interpretation of the "negative" of a vector is its spatial reflection in the 2-dimensional visual plane. A design element can be reflected across any axis in that plane. Closure under reflection then corresponds to the inclusion of all elements plus their reflections together in a structure. Design that satisfies this rule will show a collection of compound elements that possess bilateral symmetry. In practice, to do this for every possible element and its reflection would be overwhelming and redundant. Additionally, we have to choose specific axes of reflection rather than include every possible axis in each design.

An architectural "vector" grouped together with its spatial reflection creates a coupled symmetric pair, but they do not cancel each other. This is the opposite of the mathematical situation, where the summation of a vector with its negative gives the zero vector. The material situation is more complicated, since addition in the visual plane corresponds with "assembly", as smaller architectural units (vectors) are assembled into larger ones.

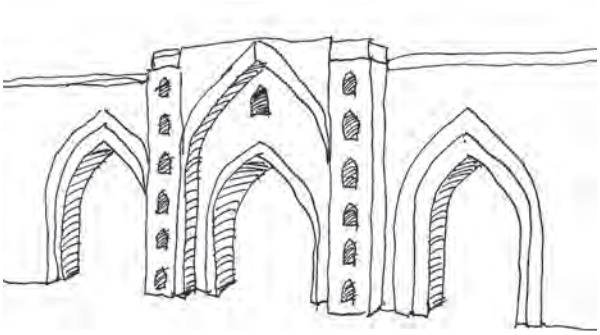

Figure 8. *Self-similar arches on different scales.*

There is a second type of "opposite" of a tectonic element that takes place in the third dimension, which is along the line of sight. It is a reflection orthogonal to the visual plane. One may define a same-size element with opposite characteristics in the depth dimension. Imagine a bas-relief compared with its negative mould. If we virtually superimpose an architectural element and its opposite in

depth, normally those should cancel out to leave nothing, i.e. a smooth flat surface. Something here is analogous to the mathematical case.

The vector space properties tell us to always have pairs of opposite elements present, and this applies in this second interpretation according to depth. If we have, say, concave and convex moldings somewhere, then, when virtually superimposed, they would cancel out to zero (flat). This application provides a working rule: *moldings should be present in equal numbers so that concave and convex parts are balanced visually*. Again, this is certainly the case with moldings in traditional and vernacular architectures, and is one reason those appeal to our innate mathematical sense.

9. Scaling similarity

Multiplication by a scalar either magnifies or reduces an architectural element without changing its internal relationships. This operation scales a figure up or down. In an architectural sense, this is a key feature in traditional architectures that include scaled-up or scaled-down copies of elements such as rectangles (door and window openings), and the curves defined in domes and arches. Those shapes are sometimes repeated further in a much scaled-down version employed in the ornamentation. As with the case of including all reflections of every architectural element, which is impossible, we cannot include all possible magnifications of elements.

Actually, it is found that scaling only occurs in a discrete hierarchy, so that the architect determines larger and smaller sizes by using a fixed scaling factor. This means that the mathematics describes not a Vector Space but a Module over the Geometric Sequence of powers of $e \approx 2.7$, $\{ e^n \}$. (The sequence is not quite an Abelian Cyclic Multiplicative Group, since there is a maximum power determined by the size of the building. Exponentials are preferable to Fibonacci numbers, which do not close under multiplication because of Carmichael's Theorem). The Module defined here requires all copies of an element scaled by the factors e^n to be present.

In practice, this process selects a discrete set of scaled copies from all possible scaled copies to include in a design. Creating a whole that is an assembly of self-similar copies defined by some scaling factor is a central feature of fractals. The architectural vectors are spatially quantized by the necessity for fractal scaling, to make them compatible with hierarchical systems. The scaling ratio of e was proposed at the beginning of this paper, which seems to satisfy a large number of traditional buildings in the architecture of many cultures around the world (Salingaros, 1998).

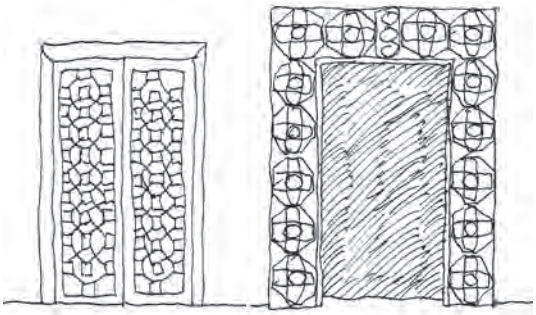

Figure 9. *Fractal scaling constrains magnifications to obey a discrete scaling factor.*

10. Vector addition generates coherence

Elements that we use to define an architectural vector space must be additive. Note that there are distinct types of addition: (*i*) alignment and assembly, and (*ii*) virtual superposition through comparison. Intuitively, creating our environment involves adding tectonic elements physically to each other, and also grouping them visually. We can perceive whether two components, either adjoining, or situated at some distance from each other, add to create a harmonious larger whole. This is the concept of addition in systems theory, where components combine to create a larger system, and is very different from the usual vector space addition.

Addition is impossible when two components clash, so that putting them together (either physically or virtually) creates a fragmented or otherwise inharmonious juxtaposition. In that case, the two elements are not additive; hence do not belong inside the same architectural space. Addition depends upon the simplest geometrical relationships: alignment, complementary fit, similar size, etc. The addition operation creates wholeness in the sense of Alexander (2001). Alexander has already detailed 15 operations that bring matter together in a way that generates more complex yet coherent larger-scale structure (Alexander, 2001; Salingaros, 2010).

One characteristic of coherent groupings created by addition through alignment and assembly is the presence of multiple symmetries. This implies that the addition operation of design vectors acts together and depends upon a process for creating symmetries. For example, addition as the assembly of complementary pairs leads to alternating repetition, a universal property documented by Alexander (Alexander, 2001; Salingaros, 2010). The corollary holds true: vectors that are not related by some type of symmetry do not add. This underlies an important connection between addition in the systemic sense and symmetry.

Addition also presupposes some measure of affinity among the constituent elements, whether they are simple or compound. Tectonic and design elements could share the same material, shape, texture, etc. Commonality makes it possible to "add" spatially-separated vectors, whereas too great a dissimilarity marks them as incompatible. If they cannot otherwise be related by another mechanism such as symmetry, then the addition operation cannot be performed, and they do not belong in the same architectural space.

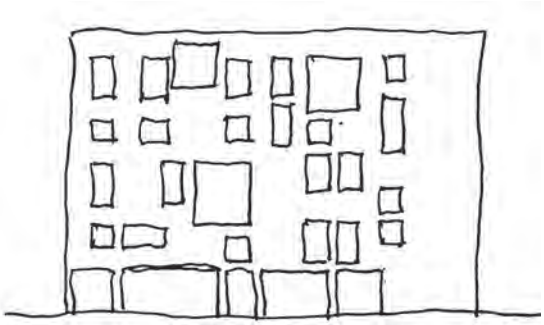

Figure 10. *Building deliberately misaligns elements to violate the architectural space.*

To summarize the addition operation, vectors add or don't add at a distance according to their symmetric relationship to each other. If they do add, they do so in a way to generate a larger element on a much larger scale as themselves. Local addition is also a process that seems to create larger wholes, as for example adding an element to its local reflection. This does not lead to cancellation, but to a coupled symmetric pair.

11. Conclusion: is beauty linked to meaning in architecture?

The brain as an information processor searches for meaning in the environment. Our evolution has prepared us to interpret patterns and classify information. Meaning thus has a biological origin directly linked to our survival and has little to do with ideology, philosophy, or politics. This is the reason why the human body reacts positively in a visceral sense to "beauty" that has healing properties, and why children respond spontaneously to such beauty.

By some accident of history that is too involved to go into here, the teaching of design has become focused on doing the opposite of the mathematical tools outlined above. Architects hold a mental model that does not seek to optimize the human-environment relationship. Fashionable design has over several decades eschewed symmetries of all types, violated gravity, and eliminated the smaller

elements that could define a fractal distribution on a structure's façade or interior (Mehaffy and Salingaros, 2013). Yet neuroscience identifies those characteristics as healing, and suggests that our body identifies them as elements of "beauty".

This conscious reversal began with an attempt at innovation through breaking from traditional practices that included mechanisms for mathematical coherence. By now this contrary approach to design has been internalized and is no longer questioned (Mehaffy and Salingaros, 2011). Attempts by architects to include design elements I mention as necessary for our sensory well-being are interpreted by the mainstream profession as violating some absolute ethical code. Theoretical explanations within architecture avoid discussing human physiology and rely instead upon a design exegesis that is reminiscent of cult beliefs (Salingaros, 2014).

Traditional and vernacular buildings aside, why is biologically-based beauty reversed in the "approved" architecture of our times? Practitioners who design the form first, as an abstraction, might feel accused and will deny that they are rejecting beauty. The same denial comes from an intellectual community that praises new and older buildings that deliberately reject the necessary mathematical constructs presented here, as well as from an educational system that has been teaching our young architects to exclusively create abstractions (Salingaros, 2017).

One reads claims that architecture is all about shaping space for habitation and movement, and is not primarily concerned with aesthetics. This statement is misleading and self-justifying, because the majority of buildings that we perceive as psychologically hostile were designed by following a very definite anti-aesthetic. Mathematical rules indeed determine what are the most comfortable volumes for each function and situation. A solid research basis for adaptive design exists, providing guidelines congruent with the mathematical rules for beauty discussed here. Countless self-builders have relied for millennia on these timeless principles for their projects. Dominant architectural culture shows no interest in this body of work, and pointedly ignores it.

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